Flow of a non-homogeneous fluid in a porous medium

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(Received 27 September 1960)

If the viscosity and specific weight of a fluid are variable, the equations governing its flow in a porous medium are non-linear and in general very difficult to solve. It has been found, however, that steady flows of a fluid of variable viscosity but constant specific weight can be reduced to those of a homogeneous fluid by a remarkably simple transformation, which indicates that the flow patterns of the fluid are the same as those of a homogeneous fluid with the same boundary conditions, and that only the speed need be modified. The speed of the actual flow is obtained by dividing the speed of the homogeneous-fluid flow by a factor proportional to the actual viscosity. The transformation is also used to derive the equations governing steady two-dimensional flows and steady axisymmetric flows of a fluid of variable viscosity and specific weight. In a good many cases of practical importance these equations are exactly linear, in spite of the fact that the governing equations obtained without the use of the above-mentioned transformation are non-linear. An exact solution for a steady two-dimensional flow with prescribed boundary conditions is given. Two inverse methods for generating exact solutions for two-dimensional flows are presented, together with two illustrative examples. The theory also applies to Hele–Shaw flows, so that it can be easily verified in the laboratory.

1. Steady seepage flow of a fluid of variable viscosity

The generalized Darcy's law for steady flows of a non-homogeneous fluid in a porous medium is expressed by the equations

$$\frac{\mu}{k}u_i = -\frac{\partial p}{\partial x_i}+\rho X_i \quad (i=1,2,3), \tag{1}$$

in which u_i is the (mean) velocity component in the direction of the Cartesian co-ordinate x_i, μ is the viscosity, k is the permeability (which may vary from place to place), p is the pressure, ρ is the density, and X_i is the body force per unit mass in the direction of increasing x_i. If the fluid is incompressible, the equation of continuity is

$$\frac{\partial u_i}{\partial x_i}=0. \tag{2}$$

If μ and ρ are constant, equations (1) and (2) are linear. In particular, if (in addition) k is constant and X_i possesses a potential Ω so that

$$X_i = -\frac{\partial \Omega}{\partial x_i}, \tag{3}$$

then

$$\phi = \frac{k}{\mu}(p+\rho\Omega)$$

is the potential for the velocity components and satisfies the Laplace equation, as can be seen from the three preceding equations. If ρ and μ are not constant, equations (1) are highly non-linear, and at first sight hopelessly complicated. The following development will show that the situation is actually not as hopeless as it appears to be.

For seepage flow with a macroscopic scale large compared with the dimension of the interstices, interstitial diffusion can be neglected† (Saffman 1959). Molecular diffusion can be neglected (Saffman 1960) if the Péclet number based on a macroscopic scale is large compared with 1. Thus in most practical cases diffusion can be neglected altogether, and for steady flows

$$u_\alpha \frac{\partial \mu}{\partial x_\alpha} = 0 \tag{4}$$

and

$$u_\alpha \frac{\partial \rho}{\partial x_\alpha} = 0. \tag{5}$$

In the absence of body forces, the effect of viscosity variation is simply and conclusively embodied in the transformation

$$u'_i = \frac{\mu}{\mu_0} u_i, \tag{6}$$

in which μ_0 is a reference viscosity and u'_i the velocity of an associated flow. To isolate the effect of viscosity variation, we shall assume ρ to be constant and the body force to be conservative. Equations (1) then become

$$\frac{\mu_0}{k} u'_i = -\frac{\partial}{\partial x_i}(p+\rho\Omega). \tag{7}$$

Furthermore, because of (4), equation (2) can be written

$$\frac{\partial u'_i}{\partial x_i} = 0. \tag{8}$$

Equations (7) and (8) are those governing the flow of a homogeneous fluid. Thus, by means of the transformation (6), the flow of a fluid of variable viscosity is related to that of a homogeneous fluid. In particular, if k is constant,

$$u'_i = -\frac{\partial \phi'}{\partial x_i}, \tag{9}$$

in which the potential

$$\phi' = \frac{k}{\mu_0}(p+\rho\Omega) \tag{10}$$

satisfies the Laplace equation

$$\nabla^2 \phi' = 0. \tag{11}$$

Consequently, if k and ρ are constant, the flow *pattern* is the same as that for a homogeneous fluid, provided the boundary conditions are unchanged.‡ The actual velocity u_i is obtained by means of (6) from the velocity u'_i of the irrotational flow

† The author owes this assurance to Dr P. G. Saffman.

‡ In this connexion, remember that the conditions at surfaces of density discontinuities are satisfied in the actual flow if they are satisfied in the associated flow (with density jumps).

field determined by (11). Thus, in regions of constant u'_i (say $u'_1 = U'$, $u'_2 = u'_3 = 0$), the actual speed u_1 is inversely proportional to the viscosity. This conclusion applies to regions of *horizontal* flow, even if ρ is not constant.

2. The equation governing steady two-dimensional flows of a non-homogeneous fluid

The effect of variation in specific weight (or of density in a gravitational field) will now be taken into account also. If (x, z) are used for (x_1, x_3), with z measured vertically upward, and (u, w) are used for (u_1, u_3), equations (1) can be written, for two-dimensional flows,

$$\frac{\mu}{k}u = -\frac{\partial p}{\partial x}, \quad \frac{\mu}{k}w = -\frac{\partial p}{\partial z} - g\rho, \tag{12}$$

or, with

$$(u', w') = \frac{\mu}{\mu_0}(u, w), \tag{13}$$

$$\frac{\mu_0}{k}u' = -\frac{\partial p}{\partial x}, \quad \frac{\mu_0}{k}w' = -\frac{\partial p}{\partial z} - g\rho. \tag{14}$$

The permeability k will be assumed constant in the subsequent development. If p is eliminated from equations (14), and, as a consequence of (5) and the steadiness of the motion, ρ is recognized to be a function of the stream function alone, the following equation is obtained:

$$\nabla^2\psi' = \frac{kg}{\mu_0}\frac{d\rho}{d\psi'}\frac{\partial \psi'}{\partial x}, \tag{15}$$

in which ψ' is the stream function (of Lagrange) for the velocity components u' and w':

$$u' = \frac{\partial \psi'}{\partial z}, \quad w' = -\frac{\partial \psi'}{\partial x}. \tag{16}$$

The quantity $d\rho/d\psi'$ is to be determined from the upstream condition.

3. The equation governing steady axisymmetric flows of a non-homogeneous fluid

For axisymmetric flows, the equations corresponding to (14) are, in cylindrical co-ordinates (with z measured vertically upward),

$$\frac{\mu_0}{k}u' = -\frac{\partial p}{\partial r}, \quad \frac{\mu_0}{k}w' = -\frac{\partial p}{\partial z} - g\rho, \tag{17}$$

in which u' and w' are again related to u and w by (13), except that now u is the radial and w the axial component of the velocity. The equation of continuity

$$\frac{\partial(ru)}{\partial r} + \frac{\partial(rw)}{\partial z} = 0 \tag{18}$$

can again be written as

$$\frac{\partial(ru')}{\partial r} + \frac{\partial(rw')}{\partial z} = 0, \tag{19}$$

because the substantial derivative of μ is zero. Equation (19) permits the use of Stokes's stream function ψ':

$$u' = -\frac{1}{r}\frac{\partial \psi'}{\partial z}, \quad w' = \frac{1}{r}\frac{\partial \psi'}{\partial r}. \tag{20}$$

Elimination of p from (17) by cross-differentiation and utilization of the fact that ρ is a function of ψ' alone produce the following equation governing steady axisymmetric flows:

$$-\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi' = \frac{kgr}{\mu_0}\frac{d\rho}{d\psi'}\frac{\partial \psi'}{\partial r}, \tag{21}$$

in which $d\rho/d\psi'$ is to be determined from the upstream condition.

4. Exact solutions for two-dimensional flows

As a first example, the case of stratified seepage into a two-dimensional sink will be discussed. The fluid is assumed to be confined to the porous layer between two impermeable horizontal planes, one at $z = 0$ and the other at $z = d$. The sink is situated in the upper plane.

The point in the lower plane directly below the sink will be used as the origin for the co-ordinates (x, z) in the plane perpendicular to the direction of the length of the sink. Thus the co-ordinates for the trace of the sink in that plane are $(0, d)$. Since there is symmetry about the z-axis, only one half of the flow field need be considered. The flow at $x = -\infty$ is, as can be verified later, horizontal in direction. Hence (15) demands that, at $x = -\infty$,

$$\psi' = Cz \quad (C = \text{constant}),$$

where ψ' is taken to be zero at the lower boundary. If the upstream variation of μ with z is given, C is related to the actual discharge in a straightforward manner. With the dimensionless variables defined by

$$\xi = \frac{x}{d}, \quad \eta = \frac{z}{d}, \quad \Psi = \frac{\psi'}{Cd}, \quad \text{and} \quad B = -\frac{kg}{\mu_0 C}\frac{d\rho}{d\Psi}, \tag{22}$$

(15) becomes

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\Psi + B\frac{d\Psi}{d\xi} = 0. \tag{23}$$

The quantity B is in general a function of Ψ, to be determined from the upstream condition. But if ρ changes linearly with z far upstream, where the flow is parallel, it also changes linearly with Ψ and B is a constant. The constancy of B will be assumed in the examples given.

Equation (23) is to be solved with the boundary conditions

$$\begin{aligned} \Psi &= 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \xi = 0 \quad (\eta < 1), \\ \Psi &= 1 \quad \text{at} \quad \eta = 1, \\ \Psi &= \eta \quad \text{at} \quad \xi = -\infty. \end{aligned}$$

The solution, by the method of separation of variables, is

$$\Psi = \eta + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi\eta \exp(\alpha\xi), \tag{24}$$

with

$$\alpha = \tfrac{1}{2}(-B + \sqrt{[B^2 + 4n^2\pi^2]}).$$

The flow pattern for $B = 0$ is an irrotational flow pattern, familiar in hydrodynamics, and will not be shown here. Those for $B = \pi$, 2π and 4π are shown in figures 1–3 respectively. The flow condition at $x = -\infty$ is entirely the same for

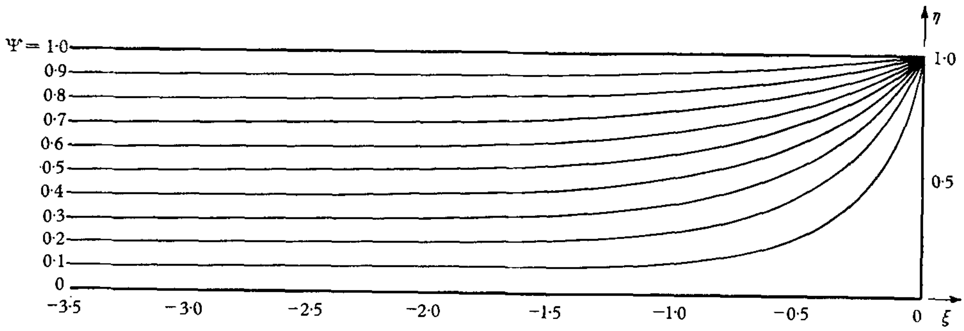

FIGURE 1. Two-dimensional flow of a stratified fluid in a porous medium into a sink, $B = \pi$.

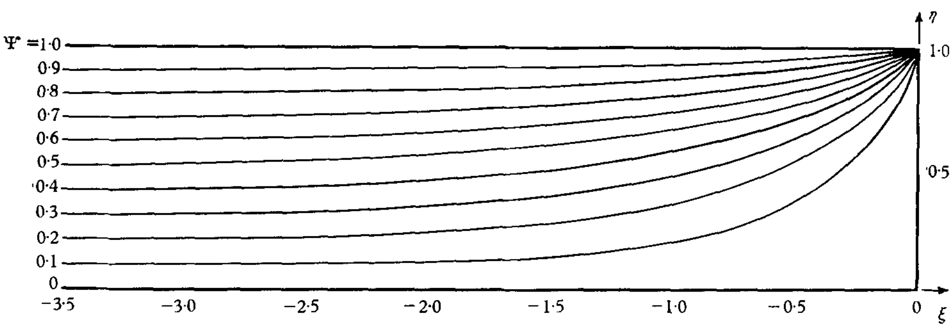

FIGURE 2. Two-dimensional flow of a stratified fluid in a porous medium into a sink, $B = 2\pi$.

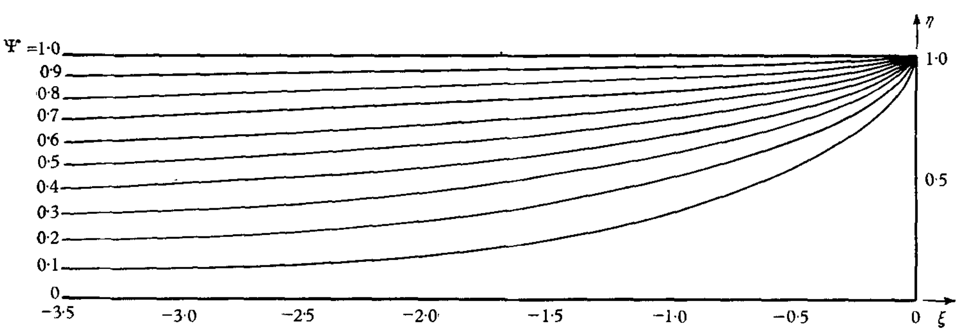

FIGURE 3. Two-dimensional flow of a stratified fluid in a porous medium into a sink, $B = 4\pi$.

any value of B (or of density variation), but the patterns show a concentration of streamlines near $\eta = 1$ for greater and greater values of B. The solution (24) and the corresponding one for axisymmetric flow may be useful for deciding whether the practice in the oil industry of forcing oil up for pumping by injecting water into the ground is an economical one.

For flows, confined between two impermeable planes $z = 0$ and $z = d$, from left to right past an impermeable barrier protruding from the lower plane (say), inverse methods can be advantageously used. Two inverse methods will now be given, which can be combined if desired. For either method the solution is of the form

$$\Psi_- = \eta + \sum_{n=1}^{\infty} A_n \sin n\pi\eta \exp(\alpha_n \xi) \quad (\xi < 0), \tag{25}$$

$$\Psi_+ = \eta + \sum_{n=1}^{\infty} B_n \sin n\pi\eta \exp(\beta_n \xi) \quad (\xi > 0), \tag{26}$$

in which

$$(\alpha_n, \beta_n) = \tfrac{1}{2}(-B \pm \sqrt{[B^2 + 4n^2\pi^2]}), \tag{27}$$

the constancy of B being assumed. The first method consists in matching Ψ_- and Ψ_+ at $\xi = 0$ through the demands

$$\Psi_- = \Psi_+ \quad \text{at} \quad \xi = 0, \tag{28}$$

$$\frac{\partial \Psi_-}{\partial \xi} - \frac{\partial \Psi_+}{\partial \xi} = f(\eta) \quad \text{at} \quad \xi = 0, \tag{29}$$

in which

$$f(\eta) = 0 \quad \text{for} \quad a \leqslant \eta \leqslant 1 \quad \text{and at} \quad \eta = 0,$$

and is arbitrary otherwise. The function $f(\eta)$ corresponds to a vortex distribution from $\eta = 0$ to $\eta = a$. Equation (28) demands that

$$A_n = B_n, \tag{30}$$

and (29) demands

$$\alpha_n A_n - \beta_n B_n = 2\int_0^1 f(\eta) \sin n\pi\eta \, d\eta, \tag{31}$$

or

$$\sqrt{(B^2 + 4n^2\pi^2)}\, A_n = 2\int_0^1 f(\eta) \sin n\pi\eta \, d\eta. \tag{32}$$

The vortex sheet does not have to be located at $\xi = 0$. If it is located at $\xi = b$, all one has to do is to change ξ to $\xi - b$ in equations (25) and (26). Thus more than one vortex sheet may be used, and the resulting barrier is then traced out as the closed streamline $\Psi = 0$. By assuming $f(\eta)$ to be a 'Dirac function' located at some non-zero elevation, equations (25)–(27), (30) and (31) provide the solution for flow over a barrier generated by a concentrated vortex located on $\xi = 0$. By shifting the origin we can obtain the solution for a concentrated vortex located elsewhere, and the solutions for different vortices can again be superposed.

If

$$f(\eta) = \begin{cases} -12 \sin 2\pi\eta & \text{in} \quad 0 \leqslant \eta \leqslant \tfrac{1}{2}, \\ 0 & \text{elsewhere}, \end{cases} \tag{33}$$

then

$$I_n \equiv 2\int_0^1 f(\eta) \sin n\pi\eta \, d\eta = \begin{cases} 0 & \text{if} \quad n \text{ is even but} \neq 2, \\ -6 & \text{if} \quad n = 2, \\ \dfrac{48}{(n^2 - 4)\pi}(-1)^{\frac{1}{2}(n-1)} & \text{if } n \text{ is odd.} \end{cases} \tag{34}$$

The flow patterns for this choice of $f(n)$ are shown in figures 4 and 5, for $B = \pi$ and 2π respectively. From (25) and (26) it can be seen that the flow pattern is always unsymmetric about $\xi = 0$, even if the barrier itself is symmetric.

The second method consists in matching Ψ_- and Ψ_+ at $\xi = 0$ by demanding

$$\frac{\partial \Psi_-}{\partial \xi} - \frac{\partial \Psi_+}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0, \tag{35}$$

$$\Psi_- - \Psi_+ = f(\eta) \quad \text{at} \quad \xi = 0, \tag{36}$$

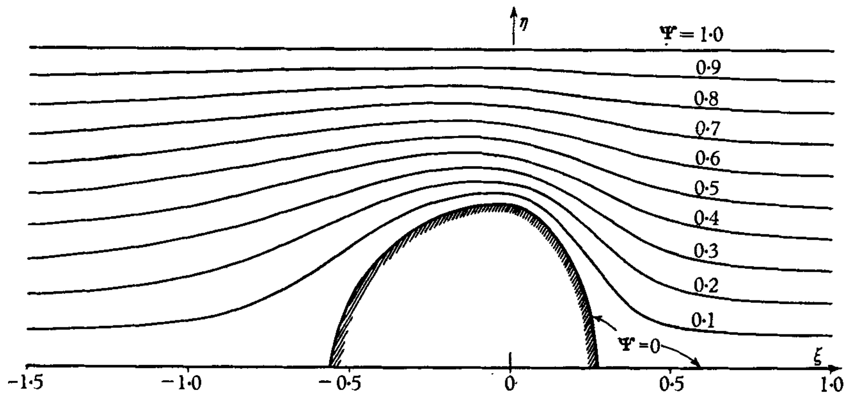

FIGURE 4. Two-dimensional flow of a stratified fluid in a porous medium over a barrier, $B = \pi$.

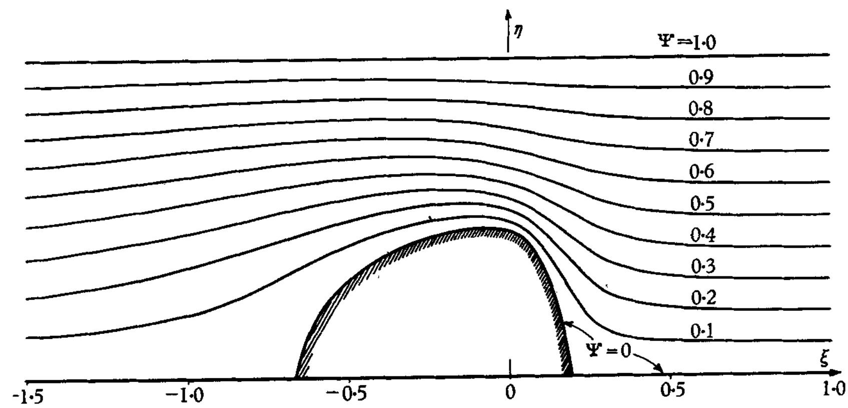

FIGURE 5. Two-dimensional flow of a stratified fluid in a porous medium over a barrier, $B = 2\pi$.

in which $f(\eta)$ is defined as in (29). The function $f(\eta)$ now corresponds to a source distribution. Equation (35) demands

$$\alpha_n A_n - \beta_n B_n = 0, \tag{37}$$

and (36) demands

$$\left(1 - \frac{\alpha_n}{\beta_n}\right) A_n = \left(2 + \frac{B}{n^2\pi^2}\beta_n\right) A_n = 2\int_0^1 f(\eta) \sin n\pi\eta \, d\eta. \tag{38}$$

Again, the source distribution may be shifted to $\xi = b$. In that case the solution is given by equations (25)–(27), (37) and (38), with ξ changed to $\xi - b$. Solutions corresponding to concentrated sources located anywhere above $\eta = 0$ can be obtained by assuming $f(\eta)$ to be a 'Dirac function', and these can be superposed on those for source distributions to obtain a solution representing a flow over a barrier which is nearly the same in form as a prescribed one. In order that a closed barrier be obtained, however, the total algebraic sum of the sources must be zero. In fact, from the solution corresponding to a concentrated source we can obtain that for a concentrated doublet, by differentiation of (25) and (26) with respect to ξ. The solution corresponding to a doublet distribution can then be obtained by integration. The doublets and doublet distributions (over vertical lines) can be located at different places, and the corresponding solutions can be superposed. We do not have to worry now about the closure of the streamline representing the barrier.

5. Stratified flow in Hele–Shaw cells

Since Hele–Shaw flows can be much more easily investigated in the laboratory and are much easier for observation than seepage flows, it is desirable to show that all of the developments in §§ 2 and 4 can be carried over to Hele–Shaw flows. In fact, if the fluid is confined between two rigid planes $y = 0$ and $y = b$, and if b is very small, the equations of motion are, for steady flows,

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}, \tag{39}$$

$$\mu \frac{\partial^2 w}{\partial y^2} = \frac{\partial p}{\partial z} + g\rho, \tag{40}$$

in which the symbols have exactly the same meanings as in § 2. Since b is small, we can assume, after Hele–Shaw, that

$$(u, w) = 6\frac{y}{b}\left(1 - \frac{y}{b}\right)(U, W),$$

where U, W are independent of y. If, furthermore, μ is assumed to be only a function of x and z, and not of y, equations (39) and (40) become

$$\frac{12\mu}{b^2} U = -\frac{\partial p}{\partial x}, \tag{41}$$

$$\frac{12\mu}{b^2} W = -\frac{\partial p}{\partial z} - g\rho, \tag{42}$$

which are identical to (12) if $\frac{1}{12}b^2$ is equated to k, and U and W identified with the u and w in (12). Since μ and ρ again do not change on a streamline in the x–z-plane, an equation identical to (15) can again be obtained, and all of the developments in § 4 can be carried over.

This work has been done at the University of Cambridge during the tenure of a Senior Post-doctoral Fellowship granted by the National Science Foundation. It is also a pleasure to acknowledge here the support of fundamental research in stratified flows by the Office of Ordnance Research in the past several years. The assistance of Mr Chintu Lai (supported by O.O.R.) in the production of the figures is much appreciated.

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